## Partial Differential Equations - Midterm exam

You have 2 hours to complete this exam. Please show all work. Each question is worth 25 points for a total of 100 points. Be sure to quote clearly any theorems you use from the textbook or class. Good luck!

(1) Show that the solution to the one dimensional heat equation in free space

$$\partial_t u(t,x) = \partial_x^2 u(t,x) \quad [0,T]\times\mathbb{R}$$
$$u(x,0) = f(x)$$

for $f(x) \in C^2(\mathbb{R})$ is

$$u(t,x) = K \star f = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} f(y)\,dy$$

Prove that $\lim_{t\to 0} K \star f = f(x)$ in your derivation.

(2) Find the solution of the wave equation in one dimensional free space with initial conditions

$$u(x,0) = \begin{cases} 0 & x \le -a \\ a^2 - x^2 & -a \le x \le a \\ 0 & a \le x \end{cases}$$
$$u_t(x,0) = 0.$$

(3) What does it mean for a partial differential equation of second order to be elliptic, parabolic, or hyperbolic?

The concept can be generalized in a straightforward way when the coefficients are functions of the variables (i.e. for quasilinear partial differential equations). Keeping in mind that in such case the equation does not necessarily fall in just one class, verify the following variant of the minimal surface equation

$$\frac{\partial^2 \varphi}{\partial x^2}(1+x^2) + \frac{\partial^2 \varphi}{\partial y^2}(1+y^2) = 2xy\frac{\partial^2 \varphi}{\partial x \partial y}$$

is elliptic. Consider the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2(x)\frac{\partial^2 \phi}{\partial x^2},$$

show that this equation is hyperbolic for $c(x) \neq 0$.

(4) State coordinates $u$,$v$ such that

$$3\frac{\partial^2 \phi}{\partial x^2} - 2\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y^2} = C\frac{\partial^2 \phi}{\partial u \partial v}$$

for some constant $C$, and find the constant $C$. Check explicitly that the equation holds for the coordinates you have chosen. Find the general solution of the equation

$$3\frac{\partial^2 \phi}{\partial x^2} - 2\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y^2} = \frac{16}{9}(x^2 - 2xy - 3y^2).$$

If we are given that

$$\phi(x,0) = \cos x, \quad \frac{\partial \phi}{\partial y}(x,0) = e^x$$

what is $\phi$?